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On the compatibility of relativistic wave equations in Riemann spaces: III

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Abstract. The massive spin-S particle equations govern spinors ξ and η of valence $n = 2S$. The number of undotted indices of these is t and $t-1$, respectively, where t is an integer $1 \leqslant t \leqslant n$. In flat space one has various mutually equivalent theories which correspond to different choices of the value of t. On account of the symmetries possessed by ξ and η the equations become mutually inconsistent for $n \geqslant 3$ when transcribed to an arbitrary Riemann space if one merely adopts minimal coupling. One therefore has the problem of constructing a separate set of non-minimally coupled equations for each relevant value of t. On a previous occasion I dealt with the case $t = n$ and I now investigate the case $t = n-1$ along analogous lines, though it is a good deal more complicated in its details. Finally the spin-$\frac{3}{2}$ and spin-2 equations are singled out for special consideration.

1. Introduction

In an arbitrary Riemannian space time V_4 of signature -2 the relativistic spin-$S(=:\frac{1}{2}n)$, non-zero mass equations

$$p^{\dot{\mu}_{s+1}}{}_{\nu_t}\xi^{\dot{\mu}_1\cdots\dot{\mu}_s\nu_1\cdots\nu_t} = \kappa\eta^{\dot{\mu}_1\cdots\dot{\mu}_{s+1}\nu_1\cdots\nu_{t-1}} \tag{1.1a}$$

$$p_{\dot{\mu}_{s+1}}{}^{\nu_t}\eta^{\dot{\mu}_1\cdots\dot{\mu}_{s+1}\nu_1\cdots\nu_{t-1}} = \kappa\xi^{\dot{\mu}_1\cdots\dot{\mu}_s\nu_1\cdots\nu_t} \tag{1.1b}$$

are mutually inconsistent (Buchdahl 1958, 1962a). In (1.1) $s+t=n \geqslant 3$, $p^{\dot{\alpha}}{}_{\beta} := \sigma^{k\dot{\alpha}}{}_{\beta}\nabla_k$, where ∇_k is the covariant derivative operator, and the field spinors are symmetric in their dotted and in their undotted indices. In a recent paper (Buchdahl 1982, hereafter referred to as II) I showed that in the special case $t = n$ the inconsistency can be removed by adding a suitable term to the right-hand member of (1.1b). Any such additional term must contain curvature quantities of the V_4 so that the modified equations are no longer 'minimally coupled'; cf § 4 of II.

A question which immediately suggests itself is this: how are the equations (1.1) to be modified to ensure mutual compatibility when $t \neq n$? Despite persistent effort I find myself unable to answer it except when $t = n-1$, i.e. $s = 1$. However, this seems to be of sufficient interest to make it worth reporting, not least because when, in particular, $S = \frac{3}{2}$ or 2 *all* alternative pairs of equations are now available which reduce to those which are in a certain sense equivalent when the V_4 is flat (Umezawa 1956).

Granted, then, that $s = 1$ the mutual incompatibility of (1.1a) and (1.1b) may be thought of as coming about by the failure, in general, of their left-hand members to be symmetric in $\dot{\mu}_{s+1}$, $\dot{\mu}_s$ and ν_t, ν_{t-1}, respectively. As in II, therefore, one may try to add appropriate spinors $\hat{\alpha}$ and $\hat{\beta}$ of valence $t+1$ on the right of equations (1.1), so chosen that consistency of the resulting equations obtains. In § 2 relations are found which have to be satisfied by these spinors, followed by an ansatz which expresses them in terms of two spinors α and β of valence $t-1$ and their derivatives. In § 3 the values of five freely disposable constants which appear here are then so chosen that one is left merely with two linear algebraic equations for α and β one of which does not contain α at all. To obtain their solution is therefore in principle merely a matter of standard manipulation; and it may therefore be regarded as known. The case $S=\frac{3}{2}$ is especially simple. In particular, α and β are simply spin vectors and their relation to ξ and η is exhibited explicitly in § 4. Of course they vanish in an Einstein space. The results obtained coincide with those obtained by pursuing the method of Fierz and Pauli (1939) when the interaction is gravitational rather than electromagnetic. Finally, in § 5 the special case $S=2$ is also set out in some detail. Here the equations for α and β are separate from each other, but their explicit solutions can be written down by inspection only when the V_4 is an Einstein space.

2. Generic modification of the equations when $s=1$

Equations (1.1) are to be modified by adding suitable terms to their right-hand members, so that the resulting equations will be mutually compatible. The additional terms must be linear in the field spinors and must vanish when the V_4 is flat. Before setting out to determine them it is highly desirable to introduce an abbreviated notation which is more explicit than that used in II and which remains unambiguous for all values of s and t. Essentially it simply represents a *set* of lower case indices by one corresponding upper case collective index. Thus the set $\dot{\mu}_1, \ldots, \dot{\mu}_s$ becomes $\dot{M}_s$ and $\nu_1, \ldots, \nu_t$ becomes N_t. Thus the spinors on the right of (1.1) are now simply written $\xi^{\dot{M}_s N_t}$ and $\eta^{\dot{M}_{s+1} N_{t-1}}$. Occasionally one may of course simply omit all indices when there is no risk of ambiguity.

When $s=1$ not more than two dotted indices ever appear, so that it is convenient here to set $\dot{\mu}_1 =: \dot{\mu}$, $\dot{\mu}_2 =: \dot{\lambda}$. The modified equations then take the form

$$p^{\dot{\lambda}}{}_{\nu_t}\xi^{\dot{\mu}N_t} = \kappa\eta^{\dot{\mu}\dot{\lambda}N_{t-1}} + \hat{\alpha}^{\dot{\mu}\dot{\lambda}N_{t-1}} \tag{2.1a}$$

$$p_{\dot{\lambda}}{}^{\nu_t}\eta^{\dot{\mu}\dot{\lambda}N_{t-1}} = \kappa\xi^{\dot{\mu}N_t} + \hat{\beta}^{\dot{\mu}N_t} \tag{2.1b}$$

where $\hat{\alpha}$, $\hat{\beta}$, both symmetric in N_{t-1}, are the as yet unknown spinors the presence of which is to ensure the mutual compatibility of these equations. On transvecting (2.1a) and (2.1b) with $\gamma_{\dot{\lambda}\dot{\mu}}$ and $\gamma_{\nu_t\nu_{t-1}}$, respectively, one has

$$p_{\dot{\mu}\nu_t}\xi^{\dot{\mu}N_t} = \gamma_{\dot{\lambda}\dot{\mu}}\hat{\alpha}^{\dot{\mu}\dot{\lambda}N_{t-1}} \qquad p_{\dot{\lambda}\nu_{t-1}}\eta^{\dot{\mu}\dot{\lambda}N_{t-1}} = \gamma_{\nu_t\nu_{t-1}}\hat{\beta}^{\dot{\mu}N_t}. \tag{2.2a, b}$$

Remove ξ and η from the left-hand members of these by means of (2.1b) and (2.1a), respectively. One thus arrives at the equations

$$\kappa\gamma_{\dot{\lambda}\dot{\mu}}\hat{\alpha}^{\dot{\mu}\dot{\lambda}N_{t-1}} + p_{\dot{\mu}\nu_t}\hat{\beta}^{\dot{\mu}N_t} = \tfrac{1}{2}\Gamma_{\dot{\lambda}\dot{\mu}}\eta^{\dot{\mu}\dot{\lambda}N_{t-1}} =: \hat{\eta}^{N_{t-1}} \tag{2.3a}$$

$$p_{\dot{\lambda}\nu_{t-1}}\hat{\alpha}^{\dot{\mu}\dot{\lambda}N_{t-1}} + \kappa\gamma_{\nu_t\nu_{t-1}}\hat{\beta}^{\dot{\mu}N_t} = \tfrac{1}{2}\Gamma_{\nu_t\nu_{t-1}}\xi^{\dot{\mu}N_t} =: \hat{\xi}^{\dot{\mu}N_{t-2}} \tag{2.3b}$$

where

$$\Gamma_{\alpha\beta} := 2S^{kl}{}_{\alpha\beta}\nabla_l\nabla_k. \tag{2.4}$$

Note that if X is any tensor-spinor, $\Gamma_{\alpha\beta}X$ is free of derivatives of X.

By setting $\hat{\alpha}$ and $\hat{\beta}$ equal to zero one of course merely recovers the conditions of compatibility $\hat{\xi}=0$, $\hat{\eta}=0$. Now (2.3) does not constitute a set of equations for $\hat{\alpha}$ and $\hat{\beta}$ because their left-hand members are only of valence $t-1$ instead of $t+1$. This suggests that one should express $\hat{\alpha}$ and $\hat{\beta}$ in terms of two spinors $\alpha^{N_{t-1}}$, $\beta^{\dot{\mu}N_{t-2}}$ of valence $t-1$. Accordingly I make the ansatz

$$\hat{\alpha}^{\dot{\mu}\dot{\lambda}N_{t-1}}=a_1p^{\dot{\lambda}(\nu_{t-1}}\beta^{\dot{\mu}N_{t-2})}+a_2p^{\dot{\mu}(\nu_{t-1}}\beta^{\dot{\lambda}N_{t-2})}+a_3\gamma^{\dot{\lambda}\dot{\mu}}\alpha^{N_{t-1}} \tag{2.5a}$$

$$\hat{\beta}^{\dot{\mu}N_t}=b_1p^{\dot{\mu}\nu_t}\alpha^{N_{t-1}}+b_2p^{\dot{\mu}(\nu_{t-1}}\alpha^{N_{t-2})\nu_t}+b_3\gamma^{\nu_t(\nu_{t-1}}\beta^{\dot{\mu}N_{t-2})}+b_4\gamma^{\nu_t(\nu_{t-1}}p^{\dot{\mu}\nu_{t-2}}p_{\dot{\rho}\sigma}\beta^{\dot{\rho}N_{t-3})\sigma} \tag{2.5b}$$

where $a_1,\dots,b_4$ are numerical coefficients whose values remain to be determined.

It should be noted that brackets enclosing a pair of indices act only on those indices whose character is the same as that of the indices immediately adjacent to the brackets and within them. Thus in $\zeta^{\alpha(\beta\dot{\mu}\dot{\nu}\gamma)}$ the brackets act only on β and γ; and there are no ambiguities as regards the specified symmetries of $\zeta^{(\dot{\mu}(\alpha\beta\dot{\nu})\gamma)}$ or of $\zeta^{\alpha(\beta[\dot{\mu}\dot{\rho}]\gamma)}$, for example.

3. Determination of α and β

The expressions (2.5) for $\hat{\alpha}$ and $\hat{\beta}$ must now be inserted into (2.3). One then has to evaluate (or better: re-express) fourteen separate terms. In this process one repeatedly needs to use the elementary identity

$$rP^{(\alpha_r}Q^{A_{r-1})}=P^{\alpha_r}Q^{A_{r-1}}+(r-1)P^{(\alpha_{r-1}}Q^{A_{r-2})\alpha_r} \tag{3.1}$$

with $Q^{A_{r-1}}$ symmetric. There is little to be gained in presenting this part of the work in full. It will suffice to illustrate what is involved by dealing with the term $p_{\dot{\lambda}\nu_{t-1}}p^{\dot{\mu}(\nu_{t-1}}\beta^{\dot{\lambda}N_{t-2})}$ in detail. Writing σ for the dummy index ν_{t-1} and using (3.1), this becomes

$$(t-1)^{-1}p_{\dot{\rho}\sigma}[p^{\dot{\mu}\sigma}\beta^{\dot{\rho}N_{t-2}}+(t-2)p^{\dot{\mu}(\nu_{t-2}}\beta^{\dot{\rho}N_{t-3})\sigma}]. \tag{3.2}$$

Now

$$p_{\dot{\rho}\sigma}p^{\dot{\mu}\sigma}=\sigma^{k\dot{\mu}\sigma}\sigma^{l}{}_{\dot{\rho}\sigma}\nabla_l\nabla_k=(\tfrac{1}{2}\delta^{\dot{\mu}}{}_{\dot{\rho}}g^{kl}+S^{kl\dot{\mu}}{}_{\dot{\rho}})\nabla_l\nabla_k=\tfrac{1}{2}(\delta^{\dot{\mu}}{}_{\dot{\rho}}\Box+\Gamma^{\dot{\mu}}{}_{\dot{\rho}}). \tag{3.3}$$

Thus the desired form of the term in question is

$$(t-1)^{-1}[\tfrac{1}{2}\Box\beta^{\dot{\mu}N_{t-2}}+\tfrac{1}{2}\Gamma^{\dot{\mu}}{}_{\dot{\rho}}\beta^{\dot{\rho}N_{t-2}}+(t-2)p_{\dot{\rho}\sigma}p^{\dot{\mu}(\nu_{t-2}}\beta^{\dot{\rho}N_{t-3})\sigma}].$$

Proceeding in this way it turns out that (2.3a) and (2.3b) become, respectively,

$$\begin{aligned}(b_1+\tfrac{1}{2}b_2)\Box\alpha^{N_{t-1}}&+[\kappa(a_1-a_2)-b_3]p_{\dot{\rho}}{}^{(\nu_{t-1}}\beta^{\dot{\rho}N_{t-2})}+\tfrac{1}{2}b_2\Gamma^{(\nu_{t-1}}{}_{\sigma}\alpha^{N_{t-2})\sigma}\\&-\tfrac{1}{2}b_4\Gamma^{(\nu_{t-1}\nu_{t-2}}p_{\dot{\rho}\sigma}\beta^{\dot{\rho}N_{t-3})\sigma}+2\kappa a_3\alpha^{N_{t-1}}=\hat{\eta}^{N_{t-1}}\end{aligned} \tag{3.4a}$$

$$\begin{aligned}\tfrac{1}{2}(ta_1+a_2)\Box\beta^{\dot{\mu}N_{t-2}}&+[(t-1)(\kappa b_1-a_3)-\kappa b_2]p^{\dot{\mu}}{}_{\nu_{t-1}}\alpha^{N_{t-1}}+\tfrac{1}{2}a_2\Gamma^{\dot{\mu}}{}_{\dot{\rho}}\beta^{\dot{\rho}N_{t-2}}\\&+\tfrac{1}{2}(t-2)a_1\Gamma^{(\nu_{t-2}}{}_{\sigma}\beta^{\dot{\mu}N_{t-3})\sigma}+(t-2)a_2p_{\dot{\rho}\sigma}p^{\dot{\mu}(\nu_{t-2}}\beta^{\dot{\rho}N_{t-3})\sigma}\\&+\kappa tb_4p^{\dot{\mu}(\nu_{t-2}}p_{\dot{\rho}\sigma}\beta^{\dot{\rho}N_{t-3})\sigma}+\kappa tb_3\beta^{\dot{\mu}N_{t-2}}=\hat{\xi}^{\dot{\mu}N_{t-2}}.\end{aligned} \tag{3.4b}$$

Now require that

$$
\begin{aligned}
&\text{(i)} && \kappa(a_1-a_2)-b_3=0 \\
&\text{(ii)} && 2b_1+b_2=0 \\
&\text{(iii)} && (t-1)(\kappa b_1-a_3)-\kappa b_2=0 \\
&\text{(iv)} && ta_1+a_2=0 \\
&\text{(v)} && (t-2)a_2+\kappa t b_4=0.
\end{aligned} \tag{3.5}
$$

Because of (i)–(iv) the first two terms on the left of each of (3.4a) and (3.4b) disappear. The choice (v) on the other hand ensures that the fifth and sixth terms on the left of (3.4b) combine into an expression free of derivatives of β:

$$(t-2)a_2(p_{\dot\rho\sigma}p^{\dot\mu(\nu_{t-2}}-p^{\dot\mu(\nu_{t-2}}p_{\dot\rho\sigma})\beta^{\dot\rho N_{t-3})\sigma}=(t-2)a_2\Gamma^{\dot\mu(\nu_{t-2}}{}_{\dot\rho\sigma}\beta^{\dot\rho N_{t-3})\sigma}$$

where

$$\Gamma^{\dot\mu\nu}{}_{\dot\rho\sigma}=2\sigma^{k\dot\mu\nu}\sigma^{l}{}_{\dot\rho\sigma}\nabla_{[l}\nabla_{k]} \tag{3.6}$$

so that, in particular, $\Gamma^{\dot\rho\nu}{}_{\dot\rho\sigma}=\Gamma^{\nu}{}_{\sigma}$. To the coefficients a_3 and b_3 arbitrary non-zero values may be assigned. If one conveniently chooses $a_3=b_3=(t+1)\kappa$ one has altogether

$$
\begin{gathered}
a_1=1 \qquad a_2=-t \qquad a_3=(t+1)\kappa \qquad b_1=t-1 \qquad b_2=-2(t-1) \\
b_3=(t+1)\kappa \qquad b_4=(t-2)\kappa^{-1}
\end{gathered} \tag{3.7}
$$

Equations (2.5) now read

$$\hat\alpha^{\dot\mu\dot\lambda N_{t-1}}=p^{\dot\lambda(\nu_{t-1}}\beta^{\dot\mu N_{t-2})}-tp^{\dot\mu(\nu_{t-1}}\beta^{\dot\lambda N_{t-2})}+\kappa(t+1)\gamma^{\dot\lambda\dot\mu}\alpha^{N_{t-1}} \tag{3.8a}$$

$$
\begin{aligned}
\hat\beta^{\dot\mu N_t}=&(t-1)\dot p^{\dot\mu\nu_t}\alpha^{N_{t-1}}-2(t-1)p^{\dot\mu(\nu_{t-1}}\alpha^{N_{t-2})\nu_t}+\kappa(t+1)\gamma^{\nu_t(\nu_{t-1}}\beta^{\dot\mu N_{t-2})} \\
&+\kappa^{-1}(t-2)\gamma^{\nu_t(\nu_{t-1}}p^{\dot\mu\nu_{t-2}}p_{\dot\rho\sigma}\beta^{\dot\rho N_{t-3})\sigma}.
\end{aligned} \tag{3.8b}
$$

Equations (2.3a) and (2.3b), on the other hand, have become

$$2\kappa^2(t+1)\alpha^{N_{t-1}}-(t-1)\Gamma^{(\nu_{t-1}}{}_{\sigma}\alpha^{N_{t-2})\sigma}=\tfrac{1}{2}\kappa^{-1}(t-2)\Gamma^{(\nu_{t-1}\nu_{t-2}}p_{\dot\rho\sigma}\beta^{\dot\rho N_{t-3})\sigma}+\hat\eta^{N_{t-1}} \tag{3.9a}$$

$$
\begin{aligned}
&\kappa^2 t(t+1)\beta^{\dot\mu N_{t-2}}+\tfrac{1}{2}(t-2)\Gamma^{(\nu_{t-2}}{}_{\sigma}\beta^{\dot\mu N_{t-3})\sigma}-\tfrac{1}{2}t\Gamma^{\dot\mu}{}_{\dot\rho}\beta^{\dot\rho N_{t-2}}-t(t-2)\Gamma^{\dot\mu(\nu_{t-2}}{}_{\dot\rho\sigma}\beta^{\dot\rho N_{t-3})\sigma} \\
&\qquad=(t-1)\hat\xi^{\dot\mu N_{t-2}}.
\end{aligned} \tag{3.9b}
$$

Inspection now reveals that α does not appear in (3.9b) at all whilst derivatives of β are also absent. In short (3.9b) is a linear algebraic equation for β. The latter may therefore be taken as known. Then, since derivatives of α do not appear in (3.9a), one has a linear algebraic equation for α and therefore α may also be regarded as known.

It is worthy of remark that the first term on the right of (3.9a) is absent not only when $t=2$, but also when $t=3$ since the Γ then annihilates the scalar $p_{\dot\rho\sigma}\beta^{\dot\rho\sigma}$. Equation (3.9) may of course be written in a more explicit form in which the curvature quantities of the V_4 appear in place of the Γ operators. It will suffice, however, to carry this through only in the special cases $S=\frac{3}{2}$ and $S=2$; see §§ 4 and 5. These are in any event of special interest since the *known* consistent equations with $s=0$ or 1 and $t+s=3$ or 4, taken together with their complex conjugates, constitute a set of equations containing all possible values of s and t consistent with $S=\frac{3}{2}$ or $S=2$.

4. Spin $\frac{3}{2}$

When $S=\frac{3}{2}$ one has to take $t=2$ in (3.8) and (3.9). Writing $\nu_1 =: \nu$ and $\nu_2 =: \rho$ for convenience, equations (3.8) at once reduce to

$$\hat{\alpha}^{\dot{\mu}\dot{\lambda}\nu} = p^{\dot{\lambda}\nu}\beta^{\dot{\mu}} - 2p^{\dot{\mu}\nu}\beta^{\dot{\lambda}} + 3\kappa\gamma^{\dot{\lambda}\dot{\mu}}\alpha^{\nu} \tag{4.1a}$$

$$\hat{\beta}^{\dot{\mu}\nu\rho} = p^{\dot{\mu}\rho}\alpha^{\nu} - 2p^{\dot{\mu}\nu}\alpha^{\rho} + 3\kappa\gamma^{\rho\nu}\beta^{\dot{\mu}}. \tag{4.1b}$$

According to (3.9) the spin vectors α^{ν} and $\beta^{\dot{\mu}}$ are to be obtained from the equations

$$6\kappa^2\alpha^{\nu} - \Gamma^{\nu}{}_{\sigma}\alpha^{\sigma} = \hat{\eta}^{\nu} \qquad 6\kappa^2\beta^{\dot{\mu}} - \Gamma^{\dot{\mu}}{}_{\dot{\rho}}\beta^{\dot{\rho}} = \hat{\xi}^{\dot{\mu}}. \tag{4.2a, b}$$

The task of eliminating the Γ operators in favour of curvature quantities is made easier by referring to an appropriate collection of identities (Buchdahl 1962b, hereafter referred to as S). Thus, for example,

$$\Gamma^{\nu}{}_{\sigma}\alpha^{\sigma} = 2S^{kl\nu}{}_{\sigma}\alpha^{\sigma}{}_{;kl} = -S^{kl\nu}{}_{\sigma}P^{\sigma}{}_{\varepsilon kl}\alpha^{\varepsilon} = -\tfrac{1}{4}R\alpha^{\nu}$$

because of S (5.4). Similarly, using also S (5.7), one finds that

$$\hat{\eta}^{\nu} = -\tfrac{1}{2}E_{mn}\sigma^{m}{}_{\dot{\mu}\sigma}\sigma^{n}{}_{\dot{\lambda}}{}^{\nu}\eta^{\lambda\dot{\mu}\sigma}.$$

In this way it turns out that equations (4.2) have the explicit solutions

$$(12m^2+R)\alpha^{\nu} = -2\sigma^{m}{}_{\dot{\mu}}{}^{\nu}\sigma^{n}{}_{\dot{\lambda}\sigma}E_{mn}\eta^{\dot{\mu}\dot{\lambda}\sigma} \tag{4.3a}$$

$$(12m^2+R)\beta^{\dot{\mu}} = -2\sigma^{m\dot{\mu}}{}_{\nu}\sigma^{n}{}_{\dot{\tau}\rho}E_{mn}\xi^{\dot{\tau}\nu\rho} \tag{4.3b}$$

with $2\kappa^2 =: m^2$. These, of course, go into each other under complex conjugation. Moreover, the minimally coupled spin-$\frac{3}{2}$, $s=1$ equations are known to be compatible in an Einstein space, that is, when $E_{mn}=0$ (Buchdahl 1958). According to (4.3) α^{ν} and $\beta^{\dot{\mu}}$ then indeed vanish. Concomitantly, the 'divergences' of ξ and η, that is, $\xi^{\nu} := p_{\dot{\mu}\rho}\xi^{\dot{\mu}\nu\rho}$ and $\eta^{\dot{\mu}} := p_{\dot{\lambda}\nu}\eta^{\dot{\mu}\dot{\lambda}\nu}$ which vanish in flat space, are now given by

$$\xi^{\nu} = 3p_{\dot{\rho}}{}^{\nu}\beta^{\dot{\rho}} + 6\kappa\alpha^{\nu} \qquad \eta^{\dot{\mu}} = 3p^{\dot{\mu}}{}_{\sigma}\alpha^{\sigma} + 6\kappa\beta^{\dot{\mu}} \tag{4.4}$$

as follows directly from (2.2) and (4.1).

Fierz and Pauli (1939) arrived long ago at compatible equations for spin-$\frac{3}{2}$ particles interacting with an electromagnetic field. One convinces oneself by inspection that their equations constitute an exact counterpart to those given in this section, electromagnetic replacing gravitational interaction. In other words, the method of Fierz and Pauli may be taken over directly into the present context ($S=\frac{3}{2}$) to give just the results obtained above.

Here it seems apposite to raise a point concerning minimal coupling. Within a given theory minimal coupling prescriptions such as 'replace ∂_k by $\partial_k - ie\phi_k$' or 'replace partial by covariant differentiation' are ambiguous: they must be supplemented by precise instructions which single out the specific equations or functionals to which they are intended to refer. As is well known, they will in general have inequivalent consequences depending on whether they are applied to a given set of first-order differential equations or directly to the second-order equations derived from them by iteration. Again, the prescription is often applied to a Lagrangian, taken to contain no derivatives of the field components higher than the first. This is done, in particular, by Fierz and Pauli; but the resulting first-order equations for the field spinors, written in a form free of the ultimately redundant auxiliary spinors, are not minimally coupled.

5. Spin 2

When $S=2$ one has to take $t=3$ (since $s=1$) and it is convenient to write $\nu_1 =: \nu$, $\nu_2 =: \rho$, $\nu_3 =: \sigma$. Then from (3.8)

$$\hat{\alpha}^{\dot\mu\dot\lambda\nu\rho} = p^{\dot\lambda(\nu}\beta^{\dot\mu\rho)} - 3p^{\dot\mu(\nu}\beta^{\dot\lambda\rho)} + 4\kappa\gamma^{\dot\lambda\dot\mu}\alpha^{\nu\rho} \tag{5.1a}$$

$$\hat{\beta}^{\dot\mu\nu\rho\sigma} = 2p^{\dot\mu\sigma}\alpha^{\nu\rho} - 4p^{\dot\mu(\rho}\alpha^{\nu)\sigma} + \gamma^{\sigma(\rho}(4\kappa\beta^{\dot\mu\nu)} + \kappa^{-1}p^{\dot\mu\nu)}p_{\dot\omega\tau}\beta^{\dot\omega\tau}). \tag{5.1b}$$

The equations (3.9) for $\alpha^{\nu\rho}$ and $\beta^{\dot\mu\nu}$ are

$$8\kappa^2\alpha^{\nu\rho} - 2\Gamma^{(\rho}{}_{\tau}\alpha^{\nu)\tau} = \hat{\eta}^{\nu\rho} \tag{5.2a}$$

$$12\kappa^2\beta^{\dot\mu\nu} + \tfrac{1}{2}\Gamma^{\nu}{}_{\tau}\beta^{\dot\mu\tau} - \tfrac{3}{2}\Gamma^{\dot\mu}{}_{\dot\omega}\beta^{\dot\omega\nu} - 3\Gamma^{\dot\mu\nu}{}_{\dot\omega\tau}\beta^{\dot\omega\tau} = 2\hat{\xi}^{\dot\mu\nu}. \tag{5.2b}$$

The Γ operator may be eliminated in favour of the curvature tensor in the usual way. One eventually finds that (5.2) become

$$(6m^2+R)\alpha^{\mu\nu} + 6C^{\mu\nu}{}_{\rho\sigma}\alpha^{\rho\tau} = -\tfrac{3}{2}E_{mn}\sigma^{m}{}_{\dot\alpha\varepsilon}\sigma^{n}{}_{\dot\beta}{}^{(\mu}\eta^{\dot\alpha\dot\beta\nu)\varepsilon} \tag{5.3a}$$

$$(6m^2+R)\beta^{\dot\mu\nu} + 4E_{mn}\sigma^{m\dot\mu\nu}\sigma^{n}{}_{\dot\rho\sigma}\beta^{\dot\rho\sigma} = -E_{mn}\sigma^{m\dot\mu}{}_{\alpha}\sigma^{n}{}_{\dot\lambda\beta}\xi^{\dot\lambda\nu\alpha\beta} - 2C^{\nu}{}_{\alpha\beta\gamma}\xi^{\dot\mu\alpha\beta\gamma}. \tag{5.3b}$$

These are separate equations for $\alpha^{\mu\nu}$ and $\beta^{\dot\mu\nu}$. To solve them is, in principle, a matter of straightforward manipulation. When the V_4 is an Einstein space the problem is trivial, for then, by inspection,

$$\alpha^{\mu\nu} = 0 \qquad \beta^{\dot\mu\nu} = -2C^{\nu}{}_{\alpha\beta\gamma}\xi^{\dot\mu\alpha\beta\gamma}/(6m^2+R). \tag{5.4}$$

In § 6 of Buchdahl (1962a) there appear modified spin-2 equations which are compatible in an Einstein space. To compare them directly with those above one has to complex conjugate them and then adopt the transcription $\xi^{\mu\nu\dot\rho\dot\sigma} \to \eta^{\dot\rho\dot\sigma\mu\nu}$, $\eta^{\mu\nu\lambda\dot\rho} \to \xi^{\dot\rho\mu\nu\lambda}$. In fact the equations so obtained do not resemble those found above.

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